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# THE INSTABILITY OF THE EQUILIBRIUM OF AN INHOMOGENEOUS FLUID IN CASES WHEN THE POTENTIAL ENERGY IS NOT MINIMAL*

V.A. VLADIMIROV

The possibility of extending the methods of proof of instability /1-3/ to the hydrodynamics of an ideal incompressible density-inhomogeneous (stratified) fluid is explored. As distinct from the general statement /3/, the rigid walls of the vessel containing the fluid are assumed to be fixed, so that the purely hydrodynamic part of the problem is isolated. Examples of a two-layer (with and without surface tension) and of a continuously stratified fluid are studied. The main result is to find Lyapunov functionals $W$ which in all cases are increasing, by virtue of the linearized equations of motion of the fluid. The structure of these functionals is such that their growth implies instability in the sense of an increase of the integrals of the disturbance-squared of the hydrodynamic fields (instability in the linear approximation in the mean square). The form of the functionals $W$ is determined by the Hamiltonian statement of the theorem on the instability of finite-dimensional mechanical systems /2/ and by the usual ways of introducing the canonical variables into the hydrodynamic problem /4, 5/. In view of the well-known equivalence of stratification and rotation effects /6, 7/, all the present results hold for two classes of rotating flows of homogeneous fluid. Lyapunov's and Chetayev's theorems (the converse of Lagrange's theorems) are well-known in analytical mechanics; they consist in proving the instability of the equilibrium position of a mechanical system when its potential energy has a maximum or a saddle point /1, 2/. The extension of these theorems to systems that contain rigid bodies and fluid is described in /3/ (Theorem III, p.178).

**1. Basic equations.** We consider the three-dimensional motions of an ideal incompressible fluid which entirely fills the domain $\tau$ with boundary $\partial\tau$. In Cartesian coordinates $x_1, x_2, x_3$ the equations of motion and the boundary conditions are

$$\rho Du_i = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial \Phi}{\partial x_i}, \quad D\rho = 0 \tag{1.1}$$

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$$\frac{\partial u_k}{\partial x_k}=0, \quad D\equiv\frac{\partial}{\partial t}+u_k\frac{\partial}{\partial x_k}$$

$$u_k n_k = 0 \text{ for } \mathbf{x}\equiv(x_1,\ x_2,\ x_3)\in\partial\tau \tag{1.2}$$

where $\mathbf{u}=(u_1, u_2, u_3)$, $\rho$, and $p$ are the velocity, density, and pressure fields, $\mathbf{n}=(n_1, n_2, n_3)$ is the normal to $\partial\tau$, and $\Phi=\Phi(\mathbf{x})$ is the potential of the external field of mass forces, such that $|\nabla\Phi|\neq 0$ in $\tau$. Summation is performed with respect to repeated subscripts. The energy integral for (1.1), (1.2) is the sum of the kinetic and potential energies:

$$T_0\equiv\int_\tau\rho\frac{u_iu_i}{2}\,d\tau, \quad \Pi_0\equiv\int_\tau\rho\Phi\,d\tau \tag{1.3}$$

$$E_0\equiv T_0+\Pi_0=\text{const}, \quad d\tau\equiv dx_1dx_2dx_3$$

The state of hydrostatic equilibrium is the solution of (1.1), (1.2) given by

$$\mathbf{u}\equiv 0, \ \rho=\rho_0(\Phi), \ p=p_0(\Phi) \tag{1.4}$$

Linearization of (1.1) at (1.4) gives the equations

$$\rho_0\mathbf{u}_t=-\nabla p'-\rho'\nabla\Phi \tag{1.5}$$

$$\rho_t'+(\mathbf{u}\nabla)\rho_0=0, \ \operatorname{div}\mathbf{u}=0$$

in which $\mathbf{u}, \rho'$, and $p'$ are the fields of velocity, density, and pressure disturbances.

2. **Two-layer fluid.** We study the stability of the hydrostatic equilibrium (1.4) of two fluids with constant densities $\rho_0=\rho_+$ and $\rho_0=\rho_-$, which fill the parts $\tau_+$ and $\tau_-$ of the domain $\tau\ (\tau=\tau_+\cup\tau_-)$. The boundary $\partial\sigma$ dividing the fluids is the same as a level $\Phi=\text{const}$. Let $\mathbf{v}$ be the unit normal to $\partial\sigma$, directed to the fluid $\rho_-$. We define on $\partial\sigma$ the function $g>0$, such that $g\mathbf{v}=\nabla\Phi$. The choice of $g>0$ means that $\Phi$ is increasing from $\tau_+$ to $\tau_-$, while everywhere on $\partial\sigma$ the force is directed from $\tau_-$ to $\tau_+$.

The linear problem of stability is studied in the class of motions with potential fields of the velocity $\mathbf{u}=\nabla\varphi$ of each fluid when there are no disturbances of the density $\rho'\equiv 0$. Description of the fluid motions amounts to considering the kinematic and dynamic conditions on the undisturbed boundary

$$N_t=u_k\nu_k, \ [\rho\varphi_t]=-[\rho]gN \tag{2.1}$$

where $N$ is the normal displacement of the fluid contact boundary; the brackets denote a jump of a quantity on $\partial\rho$: $[\varphi]\equiv\varphi_+-\varphi_-$.

Eqs.(2.1) are often obtained by linearization in Euler coordinates with "removal" of the boundary conditions from the known moving boundary $\partial\sigma^*$ to the undisturbed boundary $\partial\sigma$ /7, 8/. Difficulties then arise both in treating the fields of Euler disturbances in the domains between $\partial\sigma^*$ and $\partial\sigma$, and in the interpretation of the removal procedure. A method of linearization which avoids these difficulties and leads to the same Eqs.(2.1) is given in /9/ (Sect.13).

The analogue of the energy integral (1.3) for problem (2.1) is

$$E=T+\Pi=\text{const} \tag{2.2}$$

$$2T\equiv\rho_+\int_{\tau_+}u_ku_k\,d\tau+\rho_-\int_{\tau_-}u_ku_k\,d\tau=$$

$$\int_{\partial\sigma}[\rho\varphi]\,u_k\nu_k\,dS, \quad 2\Pi\equiv[\rho]\int_{\partial\sigma}gN^2\,dS$$

With $[\rho]g>0$, the lighter fluid is in the domain $\tau_-$, and in view of integral (2.2), we can speak of stability in the mean square. The simplest definition of stability can be given e.g., in the spirit of /10/ (p.24), by measuring the deviation of the disturbed from the undisturbed solution directly by means of the two quantities $T$ and $\Pi$. Stability of the state of rest (1.4) then implies that, given any number $\varepsilon>0$, there is a $\delta>0$ such that the satisfaction at the initial instant of $T(0)<\delta$, $\Pi(0)<\delta$ for all $t>0$, implies $T(t)<\varepsilon$, $\Pi(t)<\varepsilon$. This fact can be regarded as the hydrodynamic analogue of Lagrange's theorem on the stability of the equilibrium state when the potential energy in it is a minimum. For, it can be shown /3, 11/ that, for functional $\Pi_0$ of (1.3), its first variation vanishes by virtue of the equilibrium conditions, while the second is equal to the $\Pi$ of (2.2).

To demonstrate the instability when $[\rho]g<0$, we introduce the functional

$$W-\int_{\partial\sigma}[\rho\varphi]\,N\,dS \tag{2.3}$$

whose time derivative is, by Eqs.(2.1),

$$dW/dt = 2\,(T - \Pi) \equiv 2\,(E - 2\Pi) \tag{2.4}$$

with $T, \Pi, E$ of (2.2). Since, for $[\rho]g < 0$, we always have $\Pi < 0$, then $dW/dt > 2E$. On taking as the initial data the disturbance with $E > 0$, we obtain for (2.3) a linear estimate of the increase $W > W_0 + 2Et$. From this there follows the inequality

$$\int_{\partial\sigma} ([\rho\varphi]^2 + N^2)\,dS > 2W_0 + 4Et \tag{2.5}$$

which implies instability in the sense of an increase in the mean square values $N$ and (or) $[\rho\varphi]$. By the conditions

$$\int_{\partial\sigma} [\rho\varphi]/g\,dS = \text{const} \tag{2.6}$$

no increases in $[\rho\varphi]$ can occur due to the function, which depends on time only, which appears in the definition of the potential. Eq.(2.6) can be checked by direct calculation of the time derivative, using (2.1) and the fluid incompressibility condition in the form

$$\int_{\partial\sigma} N\,dS = 0$$

The constant in (2.6) can be made zero by a suitable choice of the constants in the definitions of $\varphi_+$ and $\varphi_-$.

Inequality (2.5) can be regarded as an analogue of Lyapunov's theorem on instability when the potential energy has a maximum in the equilibrium position (/1/, p.90).

**3. The presence of surface tension.** To simplify that treatment, we assume here that the gravity field is homogeneous, $\Phi = gx_3$, $g = \text{const} > 0$, and that the surface tension $\alpha$ on the boundary between the fluids is constant. Only two statements will be considered. In the first, the domain $\tau$ is the layer $a < x_3 < b$, so that the surfaces $\partial\tau$ and $\partial\sigma$ do not intersect. The energy integrals will have a meaning for motions which are periodic or rapidly damped with respect to $x_1, x_2$. In the second statement, the domain $\tau$ is finite, there is no surface tension on $\partial\tau$, and the surfaces $\partial\sigma$ and $\partial\tau$ interest at right angles. The angle of contact is also $\pi/2$. Under these conditions the surface $\partial\sigma$ is given by the equation $x_3 \equiv 0$. Relations (2.1) become /11/

$$N_t = u_3, \quad [\rho\varphi_t] = -[\rho]\,gN + \alpha\Delta N, \quad \Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

The kinetic energy (2.2) is unchanged, while the potential energy is

$$\Pi = \int_{\partial\sigma} \{[\rho]\,gN^2 + \alpha(\nabla N)^2\}\,dx_1\,dx_2, \quad \nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \tag{3.1}$$

For $[\rho]g > 0$ we have stability in the mean square. The difference from the case $\alpha = 0$ is that the deviation of the disturbed from the undisturbed solution is now measured by the integrals $T$ and $\Pi$, the second of which contains, not only $N$, but also its first derivatives.

The functional $W$, suitable for demonstrating instability with $[\rho]g < 0$, remains as before ((2.3), (2.4)):

$$W = \int_{\partial\sigma} [\rho\varphi]\,N\,dx_1\,dx_2, \quad dW/dt = 2\,(T - \Pi) \tag{3.2}$$

except that $\Pi$ is now taken from (3.1). The essential change is that, by virtue of the inequalities $[\rho]g < 0$, $\alpha > 0$, the integral (3.1) may either be positive definite, or have no definite sign. In the former case, we have stability of the equilibrium of the heavy fluid over the light fluid (stabilization by surface tension /11/). We can show that stability is present in the latter case by writing (3.2) as $dW/dt = 2\,(2T - E)$ and taking the initial data with $E < 0$. For this, it is sufficient to take e.g., $T(0) = 0$, $\Pi(0) < 0$. Then,

$$dW/dt > 2\,|E|, \quad W > W_0 + 2\,|E|\,t \tag{3.3}$$

after which the treatment is the same as in Sect.2.

As before, the $\Pi$ of (3.1) is the same as the second variation of the potential energy (/11, p.127). Thus, when disturbances are present with $\Pi > 0$ and $\Pi < 0$, we can say that the potential energy has neither a maximum nor a minimum in the equilibrium position. With this interpretation, the assertion about instability that follows from (3.3) can be regarded as an analogue of Chetayev's theorem (/2/, p.40).

4. Continuous stratification. The entire domain $\tau$ is now filled with inhomogeneous fluid with continuously variable density $\rho(\mathbf{x}, t)$. We consider the states of rest (1.4) with smooth functions $\rho_0(\Phi)$, $p_0(\Phi)$. For small disturbances (1.5), (1.2) the energy integral is

$$E = T + \Pi = \text{const}, \quad 2T = \int_\tau \rho_0 u_k u_k \, d\tau \tag{4.1}$$
$$2\Pi = -\int_\tau \Phi'(\rho_0)\rho^2 \, d\tau, \quad \Phi' \equiv d\Phi/d\rho_0$$

For $\Phi' < 0$, since integrals (4.1) are positive definite, the states (1.4) are stable in the mean square. This fact can also be regarded as a hydrodynamic analogue of Lagrange's theorem. Such a treatment is given in the non-linear statement in /12/.

Now, in terms of the exact Eqs.(1.1), we shall isolate the class of motions in which cases of instability can be studied. After eliminating the pressure from (1.1), we obtain the equation for the vector field $\boldsymbol{\lambda}(\mathbf{x}, t)$ to be "frozen"

$$D\boldsymbol{\lambda} = (\boldsymbol{\lambda}\nabla)\mathbf{u} \tag{4.2}$$
$$\boldsymbol{\lambda} \equiv \boldsymbol{\sigma} + \nabla\rho \times \nabla a, \quad \boldsymbol{\sigma} \equiv \operatorname{rot}(\rho\mathbf{u})$$

where $a(\mathbf{x}, t)$, by definition, satisfies the equation

$$Da = \Phi - {}^1/_2 u_i u_i + F(\rho) \tag{4.3}$$

in which $F(\rho)$ is an arbitrary function of $\rho$. The initial data $a(\mathbf{x}, 0)$ are also arbitrary. We can regard (4.2) as a generalization of the "frozen" property of the vorticity field, which follows from (4.2) with $\rho \equiv \text{const}$, and is valid for a homogeneous fluid. In this treatment, the generalizations of the potential flows are the flows with $\boldsymbol{\lambda} \equiv 0$, for which we obtain after integration, by the definition of $\boldsymbol{\lambda}$ the relation

$$\rho\mathbf{u} = \nabla\varphi + a\nabla\rho \tag{4.4}$$

in which the function $\varphi(\mathbf{x}, t)$ is taken from the conditions $\operatorname{div}\mathbf{u} = 0$ in $\tau$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\tau$.

It should be noted that (4.4) is the Clebsch form /8/ for the vector field $\rho\mathbf{u}$. The derivation of (4.4) from (1.1) shows how the Clebsch form arises naturally from integration of the equations of motion of an inhomogeneous fluid. Clearly, (4.4) holds, for example, if the motion arises from the state of rest, or more widely, from the state with $\boldsymbol{\sigma}(\mathbf{x}, 0) \equiv 0$. For, in view of the arbitrariness in the choice of $a(\mathbf{x}, 0)$, we can take $a(\mathbf{x}, 0) \equiv 0$, and the equation $\boldsymbol{\lambda}(\mathbf{x}, t) \equiv 0$, see (4.2), will be satisfied.

The second restriction on the class of motions consists of the fact that the Lagrangian disturbance of the denisty field vanishes. In other words, the initial data $\rho(x, 0)$ of the disturbed motion are obtained by a variation of the coordinate of each fluid particle while leaving its density unchanged. It is easy to write this with the aid of the field of Lagrangian displacements $\boldsymbol{\xi}(\mathbf{x}, t)$ /9/.

In the light of these two restrictions, the final system of equations of motion, linearized at (1.4), can be written as

$$a_t = -\Phi'(\rho_0)\rho, \quad \rho = -(\boldsymbol{\xi}\nabla)\rho_0, \quad \boldsymbol{\xi}_t = \mathbf{u}, \quad \operatorname{div}\boldsymbol{\xi} = 0 \tag{4.5}$$

with the boundary conditions on $\partial\tau$

$$\boldsymbol{\xi}\cdot\mathbf{n} = 0 \tag{4.6}$$

and the initial conditions

$$a(\mathbf{x}, 0) = 0, \quad \boldsymbol{\xi}(\mathbf{x}, 0) = \boldsymbol{\xi}_0(\mathbf{x}) \tag{4.7}$$

where the first equation in (4.5) is obtained after linearizing (4.3) and taking $F(\rho) = -\Phi(\rho)$.

From $a(\mathbf{x}, 0) = 0, \rho_0\mathbf{u}(\mathbf{x}, 0) = \nabla\varphi, \operatorname{div}\mathbf{u} = 0$ in $\tau$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\tau$, it follows that $\mathbf{u}(\mathbf{x}, 0) \equiv 0$ (/13/, p.112). For the field $a(\mathbf{x}, t)$ it follows from (4.5)-(4.7) that

$$dI/dt = 0, \quad I = 0; \quad I \equiv \int_\tau a \, d\tau \tag{4.8}$$

To demonstrate the instability of states (1.4) when $\Phi' > 0$, we now take the functional $W$ in the form

$$W = \int_\tau \rho a \, d\tau \tag{4.9}$$

Using (4.5)-(4.7) to calculate the time derivative, we obtain

$$dW/dt = 2(T - \Pi) \equiv 2(2T - E) \tag{4.10}$$

with the $T, \Pi$, and $E$ of (4.1). If we have $\Phi' > 0$ everywhere in $\tau$, then $\Pi < 0$. Since in this class of motions $T(0) = 0$, then $E < 0$ and we obtain from (4.10) a linear estimate of the increase $W > W_0 + 2|E|t$. It is now clear that

$$\int_\tau (\rho^2 + a^2)\, d\tau > 2W_0 + 4|E|t \tag{4.11}$$

In accordance with (4.11), the mean square values of $\rho$ and $a$ are increasing. The growth of the density disturbances in fact signifies physically instability. At the same time, the growth of $a$ may occur as a result of the function of time which appears additively in it, and may not signify an increase in velocity. This possibility is excluded by the equation $I = 0$, see (4.8).

Now assume that the function $\rho_0(\Phi)$ is not monotonic. There are then in $\tau$ domains of both increasing and decreasing density in the direction of the force vector. For smooth $\rho_0(\Phi)$ there always exist values of $\Phi$, at which $d\rho_0/d\Phi = 0$, so that (4.1) and (4.5) become meaningless. However, for disturbances with $\rho = -(\xi\cdot\nabla)\rho_0$, the potential energy (4.1) and the first of Eqs.(4.5) may be rewritten in forms which have no singularities:

$$2\Pi = \int_\tau \frac{\partial\Phi}{\partial x_k}\xi_k\rho\, d\tau, \quad a_t = \frac{\partial\Phi}{\partial x_k}\xi_k$$

In the present case, there are disturbances both for $\Pi > 0$ and for $\Pi < 0$ in the neighbourhood of the equilibrium position. If we choose the initial data with $T(0) = 0$, $\Pi(0) < 0$, we again arrive at (4.9)-(4.11).

Recalling that $\Pi$ is the second variation of the potential energy $\Pi_0$ of (1.3), we can interpret inequality (4.11) as the linear hydrodynamic analogue of the Lyapunov and Chetayev theorems on instability.

**5. On the choice of the functional $W$.** The heuristic basis for choosing the functionals $W$ of (2.3), (4.9) is the analogy with finite-dimensional Hamiltonian systems. If $q_i, p_i$ $(i = 1, 2, \ldots n)$ are the generalized coordinates and momenta of this system, then the Lyapunov function in the proof of instability has the form $W = p_i q_i$ (/2/, p.40). Expressions (2.3) and (4.9) are similar in structure, apart from replacement of summation over the degrees of freedom by integration. For, in the Hamiltonian statements of the hydrodynamics of a two-layer fluid, the role of canonical variables is played by $[\rho\varphi]$ and $\xi\cdot\mathbf{v}$ /4/, or for an inhomogeneous fluid, by $a$ and $\rho$ /5/.

The key role of our restrictions on the classes of motions must be specially mentioned. Notice that, with the aid of the field of Lagrangian displacements $\xi$, the functional $W$ (2.3), (3.2) can be written as the sum of volume integrals

$$W = \rho_+ \int_{\tau_+} \mathbf{u}\xi\, d\tau + \rho_- \int_{\tau_-} \mathbf{u}\xi\, d\tau \tag{5.1}$$

and can be understood more widely, by assuming that the velocity field is rotational ($\omega \equiv \operatorname{rot}\mathbf{u} \neq 0$), and satisfies in $\tau_\pm$ the equations

$$\rho_\pm \mathbf{u}_t = -\nabla p, \quad \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u} = \xi_t \tag{5.2}$$

on $\partial\tau$ the no-flow conditions

$$\mathbf{u}\cdot\mathbf{n} = \xi\cdot\mathbf{n} = 0 \tag{5.3}$$

and on $\partial\sigma$ the kinematic and dynamic conditions

$$N_t = \xi_t\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v}, \quad [p] = [\rho]\,\xi\cdot\nabla\Phi \tag{5.4}$$

which can be obtained by the method described in /9/ (see the remarks on (2.1)).

For the functional (5.1), we obtain by (5.2)-(5.4) without assuming that the disturbances are potential, the equation $dW/dt = 2(T - \Pi)$, with $T$ and $\Pi$ from (2.2), the kinetic energy $T$ being taken only in the form of a sum of volume integrals. With $\Pi < 0$, we have for (5.1): $W > W_0 + 2Et$, which leads, instead of (2.5), to the inequality

$$\rho_+ \int_{\tau_+} (|\mathbf{u}|^2 + |\xi|^2)\, d\tau + \rho_- \int_{\tau_-} (|\mathbf{u}|^2 + |\xi|^2)\, d\tau > 2W_0 + 4Et \tag{5.5}$$

from which the mistaken conclusion can be drawn about instability in the mean square norm with respect to the velocities and displacements. The point is that, here, due to the absence in an ideal fluid of velocity displacement damping, the displacements of the fluid particles in general increase linearly even in the case of stable density stratification. Thus a linear growth of (5.1) does not always correspond to the physical fact of instability.

For explanation, we take the rotational flows in the domains $\tau_+$ and $\tau_-$ when stable density layering $[\rho]g > 0$ is present. It follows from (5.2) that the vortex field $\omega_t \equiv 0$ is stationary, and no further restrictions are imposed on $\omega(\mathbf{x})$. On taking certain vortex fields $\omega_\pm(\mathbf{x})$ in $\tau_\pm$ and imposing the condition $N = \xi \cdot \mathbf{v} = 0$ on $d\sigma$, we can construct in each of domains $\tau_\pm$ stationary solutions $\mathbf{u}_\pm(\mathbf{x})$ in which always $p_\pm \equiv 0$. Conditions (5.3) and (5.4) are then satisfied. For these solutions $\xi = \xi_0(\mathbf{x}) + \mathbf{u}(\mathbf{x})t$, and we have linear growth of the functional (5.1) and quadratic growth of (5.5). At the same time, the present disturbance is stationary, and of course, there is not instability.

In short, the restriction of the class of potential disturbances, made in Sects.2 and 3, is important in principle. It is for potential disturbances that the functional (5.1) reduces to forms (2.3), (3.2), which enable us to consider only normal displacements of the dividing surface, and not arbitrary displacements of the fluid particles.

The same can be said of functional (4.9) and the class of motions (4.5). By using (4.4) and (4.5), we can reduce (4.9) to the form

$$W = \int_\tau \rho_0 \mathbf{u} \cdot \xi \, d\tau$$

For disturbances which cannot be written in the form (4.4), a linear growth of the functional does not signify instability. Here, even in the case of stable stratification $\Phi' < 0$, the components of $\xi$ parallel to the surfaces $\rho_0 = \text{const}$ can increase linearly.

*Notes*. 1°. When writing the integrals of the sums of squares of disturbances (2.5), (4.11) (5.5), it is understood that dimensionless variables are introduced for each statement. As the scales of length and time, we can in all cases take the vessel dimensions $L$ and the quantity $(L/g)^{1/2}$.

2°. By using the equivalence of the effects of stratification and rotation as described in /6, 7/ (Chapter 8), we can obtain directly by a change of notation assertions about the instability of two classes of rotating flows of homogeneous fluid. These include translationally invariant and rotationally symmetric flows. An example of the latter is Couette flow between rotating cylinders when the square of the velocity circulation decreases as the radius increases. To obtain the relevant statements, it is sufficient to take the equivalents of the density and gravity field of /6, 7/ and use the relations of the present paper.

3°. From the mathematical point of view, our assertions about instability are in the nature of a priori estimates, since no suitable theorems of existence are proved for the solutions.

4°. Since our results are of a linear kind, they can be related to the conclusions of spectral theory.

5°. The merits of our approach are simplicity, generality, the similarity of its ideas to the mechanics of finite-dimensional systems, and the fact that it is possible in principle to consider non-linear statements.

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# THREE-DIMENSIONAL DISTURBANCES IN A COMPRESSIBLE BOUNDARY LAYER*

I.V. SAVENKOV

The propagation of three-dimensional disturbances from impulsive and harmonic sources in a compressible boundary layer on a plane plate is discussed. It is assumed that the Reynolds number tends to infinity. The field of the perturbed motion is obtained in the context of the linearized theory of the boundary layer with selfinduced paressure. The solution of the linearized equations is decomposed into Fourier integrals. When finding the inverse transformations, numerical and asymptotic methods are combined. A comparison is made with experimental data and calculations of the linearized Navier-Stokes equations. The theory of a boundary layer (BL) with selfinduced pressure /1, 2 / is useful for studying the BL instability in an incompressible fluid at high Reynolds numbers $R$, see e.g., /3-9/. At the same time, the asymptotic theory /1, 2/ predicts stability (in the limit as $R \to \infty$) of the supersonic BL with respect to plane disturbances propagating strictly along the flow, which is inconsistent with the well-known results for finite $R$, see e.g., /10, 11/. In the framework of asymptotic theory, however, the supersonic BL is unstable with respect to oblique waves (travelling at non-zero angles to the incoming flow) /12, 13/. It can therefore be expected that the packet of oblique waves of instability in the limit as $R \to \infty$ is qualitatively correctly described by the theory /1, 2/. (All the more, because at finite $R$, as the Mach number $M_\infty$ of the incoming flow increases, the oblique waves become the most unstable, and their role is significantly increased in the supersonic BL /10, 11/). A packet of oblique waves is generated by any source which introduces serious three-dimensional perturbations into the boundary layer. In the present paper, such a source is taken to be injection and extraction via holes in the plate. The solution of specific problems assumes a detailed analysis of the influence of the Mach number $M_\infty$ on the BL stability (in the limit as $R \to \infty$).

## 1. Time instability.

We start by analysing the dispersion relation (DR)

$$F(\Omega, k, m; M_\infty) = \Phi(\Omega) - Q(k, m; M_\infty) = 0 \tag{1.1}$$

$$\Phi = \frac{d\,\mathrm{Ai}(\Omega)}{d\zeta} I^{-1}(\Omega), \quad I = \int_\Omega^\infty \mathrm{Ai}(\zeta)\,d\zeta, \quad \Omega = \omega (ik)^{-2/3}$$

$$Q = (ik)^{1/3}(k^2 + m^2)/\sqrt{S}, \quad S = m^2 + (1 - M_\infty^2)k^2$$

obtained after linearizing the equations of the freely interacting compressible BL /14/ with respect to disturbances of the type $f(y)\exp(\omega t + ikx + imz)$ /3, 12, 13/ ($x, y, z$ are dimensionless coordinates, measured respectively downstream, along the normal to the plate, and in the lateral direction, $t$ is dimensionless time, $M_\infty$ is the Mach number of the incident uniform flow, and $\mathrm{Ai}(\zeta)$ is the Airy function. In the case $S < 0$ we understand by the root

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